Dual Dig Level I (2011) - Solutions

1. Pat is thinking of a positive integer. The sum of its square and its (positive) square root is 84. What integer is Pat thinking of?

Explanation: By trial-and-error, we can find that the answer is 9: $(9)^2 + \sqrt{9} = 81 + 3 = 84$. (Possibilities can be quickly eliminated. We know that the answer could not be 10 or greater, because its square would then be 100 or greater, which would already exceed 84. Also, in order for its positive square root to be an integer, the answer must be a perfect square, and the only positive perfect squares less than 10 are 1, 4, and 9.)

Answer: 9

2. A quadrilateral *ABCD* has four interior angles measured in degrees. Angle *B* measures 20° more than Angle *A*. Angle *C* measures 20° more than Angle *B*. Angle *D* measures 20° more than Angle *C*. What is the measure of Angle *A*?

Explanation: Let n = the measure of Angle *A* in degrees. Then, the other angles have measures $n+20$, $n+40$, and $n+60$ in degrees. The sum of the [measures of the] interior angles of a quadrilateral is 360°. (Imagine the quadrilateral being divided into two triangles.) We have: $n+(n+20)+(n+40)+(n+60)=360 \Leftrightarrow 4n+120=360 \Leftrightarrow n=60^\circ$

Answer: 60°

3. You have three separate boxes and inside each box are two separate smaller boxes, and inside each of these smaller boxes are four even smaller boxes. What is the total number of boxes?

Explanation: There are 3 large boxes, 6 smaller boxes, and 24 smallest boxes. 3+6+24=33. Note: a single box has 11 boxes inside (including itself).

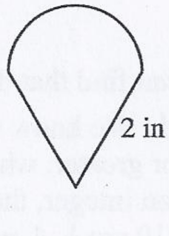
Answer: 33 total boxes

4. Simplify: $(x+1)+(x-2)+(x+3)+(x-4)+...+(x+99)+(x-100)$

Explanation: 1st, note that there are 100 groups (terms), so there are 100 "x"s.
2nd, note that there are 50 consecutive pairs of groups (terms) for which the sum of the constants is –1. The sum of all of the constants must be 50(–1), or –50.

Answer: $100x - 50$

5. A first-grade student daydreams about an ice cream cone and (carefully) draws the figure below (not to scale). The figure consists of a semicircle and two sides of an equilateral triangle of side length two inches, with the missing side of the triangle serving as the diameter of the semicircle. What is the perimeter of the figure?



Explanation: The diameter of the semicircle is two inches, so its radius is one inch. Therefore, its length is $\frac{1}{2}(\text{circumference of full circle}) = \frac{1}{2}(2\pi r) = \pi r = \pi(1) = \pi$ inches.

The two drawn sides of the triangle have total length: $2 + 2 = 4$ inches.
Therefore, the perimeter of the figure is $(\pi + 4)$ inches.

Answer: $(\pi + 4)$ inches.

6. Find all real solutions of the equation: $\frac{x}{2} + \frac{2}{x} = \frac{x}{3} + \frac{3}{x}$

Explanation 1: Multiply both sides of the equation by the LCD, $6x$.

$$6x\left(\frac{x}{2} + \frac{2}{x}\right) = 6x\left(\frac{x}{3} + \frac{3}{x}\right) \Leftrightarrow 3x^2 + 12 = 2x^2 + 18 \Leftrightarrow x^2 = 6 \Leftrightarrow x = \pm\sqrt{6}$$

Explanation 2: Subtract $\frac{2}{x}$ and $\frac{x}{3}$ from both sides of the equation so that the x terms are isolated on one side and the $1/x$ terms are isolated on the other side.

$$\frac{x}{2} - \frac{x}{3} = \frac{3}{x} - \frac{2}{x} \Leftrightarrow \frac{3x}{6} - \frac{2x}{6} = \frac{1}{x} \Leftrightarrow \frac{x}{6} = \frac{1}{x} \Leftrightarrow x^2 = 6 \Leftrightarrow x = \pm\sqrt{6}$$

Answer: $\{-\sqrt{6}, \sqrt{6}\}$.

7. For the positive integer n, let <*n*> denote the sum of all positive divisors of n with the exception of n itself. For example, <4> = 1 + 2 = 3, and <12> = 1 + 2 + 3 + 4 + 6 = 16. What is <<< 6 >>> ?

Explanation: <6> = 1 + 2 + 3 = 6, so <<6>> is 6, and <<<6 >>> is 6.

Answer: 6

8. Find the <u>inverse</u> of the function f, where: $f(x)=\dfrac{2x-5}{x+6}$

Explanation: 1[st]: Replace $f(x)$ with y, and switch $x \leftrightarrow y$: $x=\dfrac{2y-5}{y+6}$

2[nd]: Cross multiply: $xy+6x=2y-5$
3[rd]: Solve for 'y':

$$xy-2y=-5-6x \Leftrightarrow y(x-2)=-5-6x \Leftrightarrow y=\frac{-5-6x}{x-2} \Leftrightarrow y=\frac{6x+5}{2-x}$$

Answer: $f^{-1}(x)=\dfrac{6x+5}{2-x}$

9. Suppose $f(x,y)$ is some function such that: $f(x,y)=3x^2-2y^2+xy$.
Find: $f\left(f(1,2),\ f(2,1)\right)$

Explanation: 1[st] : $f(1,2)=3(1)^2-2(2)^2+(1)(2)=-3$
2[nd]: $f(2,1)=3(2)^2-2(1)^2+(2)(1)=12$
3[rd]: $f(-3,12)=3(-3)^2-2(12)^2+(-3)(12)=-297$

Answer: -297

10. There exists a triangle that has the points $(0,0),(1,0)$, and $(1,\sqrt{3})$ as its vertices. What is the mean of the distances from each vertex to the midpoint of the hypotenuse?

Explanation: Let $A=(0,0), C=(1,0)$, and $B=(1,\sqrt{3})$. The midpoint (M) of $\overline{AB}$ = $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, and, using the distance formula, the lengths are: $AM=1$, $BM=1$, and $CM=1$. (Also, observe that $AM+BM=AB=2$, because the triangle is 30°-60°-90°, and the legs have length 1 and $\sqrt{3}$.) Thus, the average (mean) of the three lengths is also 1.

Answer: 1

11. Four roommates are all late for their Calculus Final Exam. They rush into the classroom and tell the teacher, "We came to school in the same car and had a flat tire, and by the time we changed it, we'd missed the Final." The teacher put each student in a different corner of the room, gave each one a single piece of paper, and asked them to indicate which tire went flat. The teacher picked up the papers after 15 seconds. What is the probability that they all selected the same tire, assuming their selections were random? (Disregard the 'spare tire.')

Explanation: The first student selects a tire (say, Right Rear). Then, the real question becomes: 'What is the probability that the other three students match the first student's answer?'

Using the Multiplication Principle of probability:
$P(1^{st}$ Student matches$)\cdot P(2^{nd}$ Student matches$)\cdot P(3^{rd}$ Student matches$)\cdot$

$P(4^{th}$ student matches$) = (1)\cdot\left(\frac{1}{4}\right)\cdot\left(\frac{1}{4}\right)\cdot\left(\frac{1}{4}\right)=\frac{1}{64}$

Answer: $\frac{1}{64}$

12. For some unknown positive base x, the following are true: $\log_x 3 = A$, and $\log_x 2 = B$. Express $\log_x(0.75x^3)$ in terms of A and/or B.

Explanation: Since $0.75=\frac{3}{4}=\frac{3}{2^2}$, then:

$$\log_x(0.75x^3)=\log_x(0.75)+\log_x\left(x^3\right)\ \text{(by Product Rule for Logs)}$$
$$=\log_x\left(\frac{3}{2^2}\right)+\log_x\left(x^3\right)$$
$$=\log_x 3-\log_x\left(2^2\right)+3$$
$$=\log_x 3-2\log_x 2+3$$
$$=A-2B+3$$

Answer: $A-2B+3$

13. Simplify the fraction completely: $\dfrac{\left(3^{2011}\right)^2-\left(3^{2009}\right)^2}{\left(3^{2010}\right)^2-\left(3^{2008}\right)^2}$

Explanation: 1st – (for simplicity) let $x = 2008$.

Then, we have: $\dfrac{\left(3^{x+3}\right)^2-\left(3^{x+1}\right)^2}{\left(3^{x+2}\right)^2-\left(3^{x}\right)^2}=\dfrac{\left(3^{2x+6}\right)-\left(3^{2x+2}\right)}{\left(3^{2x+4}\right)-\left(3^{2x}\right)}=\dfrac{\left(\dfrac{3^{2x+6}}{3^{2x}}\right)-\left(\dfrac{3^{2x+2}}{3^{2x}}\right)}{\left(\dfrac{3^{2x+4}}{3^{2x}}\right)-\left(\dfrac{3^{2x}}{3^{2x}}\right)}$; hint: subtract the exponents. Now, $\dfrac{3^6-3^2}{3^4-1}\rightarrow(\mathit{factor})\rightarrow\dfrac{3^2(3^4-1)}{3^4-1}=3^2=9$.

Answer: 9

14. Kenny uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If the coin comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Kenny's sequence is an integer?

Explanation: Use a tree diagram and apply the rules:
1st term = 6;
2nd term is 11 or 2;
3rd term is 21, 4.5, 3, or 0;
4th term is 41, 9.5, 8, 1.25, 5, 0.5, -1, or -1.
Of the eight (equally likely) potential 4th terms, five are integers, so the correct answer is 5/8.

Answer: 5/8

15. If $\dfrac{1}{x^3}-\dfrac{1}{x^2}-\dfrac{1}{x}-1=0$, then what is the value of x^3+x^2+x+1?

Explanation: Multiply both sides of the given equation by the LCD, x^3.
We obtain: $1-x-x^2-x^3=0$.
If we add x^3, x^2, and x to both sides, we obtain: $1=x^3+x^2+x$.
If we add 1 to both sides (so that the right-hand side is the expression we want to evaluate), we obtain: $2=x^3+x^2+x+1$.

Answer: 2

16. Circle #1 and Circle #2 are concentric circles ("concentric" means "sharing the same center"). Circle #1 has radius *x*. Circle #2 is a larger circle drawn outside of Circle #1 so that the area of the ring (the region outside of Circle #1, but inside of Circle #2) has exactly the same area as Circle #1. Find a simplified expression for the radius of Circle #2.

Explanation: Let x = radius of Circle #1, y = width of ring, so $(x+y)$ = radius of Circle #2.
Area of Circle #1 = πx^2; Area of Circle #2 = $\pi(x+y)^2$.

Thus, Area of Ring = $\pi(x+y)^2 - \pi x^2$.
Remembering that Area of Ring = Area of Circle #1, we obtain:
$\pi x^2 = \pi(x+y)^2 - \pi x^2 \Leftrightarrow 2\pi x^2 = \pi(x+y)^2 \Leftrightarrow 2x^2 = (x+y)^2 \Leftrightarrow x\sqrt{2} = (x+y)$.
Finally, since $(x+y)$ represents the radius of Circle #2, ...

Answer: $x\sqrt{2}$

Challenge / Extension: Repeat this process by considering a sequence of larger concentric circles such that the outermost rings all have the same area as Circle #1. Observe the pattern in the corresponding radii.

17. In the expansion of $(x^5+x)^{100}$ written in descending powers of *x*, what is the coefficient of x^{496}?

Explanation 1: Think of $(x^5+x)^{100}$ as $(x^5+x)(x^5+x)\cdots(x^5+x)$. If we apply the Distributive Property repeatedly without combining like terms, we end up with 2^{100} terms. Each term is obtained by taking either x^5 or x from each of the 100 factors and multiplying those 100 "choices" together. An x^{496} term is obtained if and only if x^5 is "chosen" from 99 of the factors and x is chosen from the remaining factor. There are 100 ways to do this, since the x can be chosen from any of the 100 factors. Therefore, there are 100 x^{496} terms, and we obtain $100x^{496}$ after we combine like terms.

Explanation 2 (Binomial Theorem): Let $a = x^5$ and $b = x$. We want the coefficient of $a^{99}b$, which corresponds to $(x^5)^{99}x = x^{496}$. By the Binomial Theorem, this coefficient is given by: $\binom{100}{99} = \binom{100}{1} = 100$, or ${}_{100}C_{99} = {}_{100}C_1 = 100$.

Answer: 100

18. Convert 523 (base 7) to a base 5 numeral.

Explanation: 523 (base 7) is equivalent to: $5(7^2) + 2(7^1) + 3(7^0)$; which equals: 262. Note: when we do not indicate the base, it is assumed to be a 'normal' everyday base 10.

Now, in base 5, we are interested in the powers of 5: That is: $5^3 = 125$, $5^2 = 25$, $5^1 = 5$, and $5^0 = 1$

So, by deconstructing 262 into groups of powers of 5, we obtain: 2(125) + 0(25) + 2(5) + 2.

Thus, 523 (base 7) = 262 = 2022 (base 5)

Answer: 2022 (base 5)

19. Pat and Riley work for the same company. Pat says, "19/40 of my co-workers are female." Riley says, "12/25 of my co-workers are female." (a) How many total workers are there (including Pat and Riley). (b) Is Pat female? (Yes or No). Is Riley female? (Yes or No).

Explanation: The LCM of 40 and 25 is 200. The only positive integer common multiples of 40 and 25 are multiples of 200. The answer to a) must be of the form $200n+1$, where n is a positive integer. Pat then has $200n$ co-workers, as does Riley. Assume $n=1$ for now, meaning that there are 201 workers, and Pat and Riley both have 200 co-workers. Then, Pat has 95 female co-workers, since (19/40)(200) = 95. Riley has 96 female co-workers, since (12/25)(200) = 96. This is all possible, provided that Pat is female and Riley is not. If $n=2$, then Pat would have 190 female co-workers, and Riley would have 192 female co-workers, but that would not be possible; similarly, n could not be 3, 4, 5, ….

Answer: a) 201 workers, b) Pat: Yes, Riley: No

20. Suppose you write out all the integers from 1 to 1000 inclusive. How many times would you write the digit '1'?

Explanation 1: Envision the integers from 1 to 99 with leading '0's so that we obtain 001 to 099. (We may also include 000 for consistency purposes.) There are 10 such integers with a '1' in the ones place: 001, 011, …, 091. There are 10 such integers with a '1' in the tens place: 010, 011, …, 019. Observe that '011' is appropriately double-counted. There are 20 '1's in the tens and ones places in the integers from 000 to 099. Consider the following ten blocks of integers: 000 to 099, 100 to 199, 200 to 299, …, 900 to 999. Each of the ten blocks has 20 '1's in the tens and ones places, so there are a total of 200 '1's in the tens and ones places. In addition, there are 100 '1's in the hundreds place in the integers 100 through 199, and there is also the one '1' in the thousands place of 1000. Therefore, the answer is 200+100+1=301.

Explanation 2 (Case analysis by number of digits):
• 1-digit case: '1' has one '1'.
• 2-digit case: The ten integers from 10 through 19 have a '1' in the tens place. There are nine integers (11, 21, 31, …, 91) that have a '1' in the ones place. Note that '11' is appropriately double-counted. Therefore, the 2-digit case yields 19 '1's, because 10+9=19.
• 3-digit case: The 100 integers from 100 through 199 have a '1' in the hundreds place. If we discard the '1' in the hundreds place, we refer "recursively" to the 1-digit (01-09) and 2-digit (10-99) cases above and obtain an additional 20 '1's in the tens place and in the ones place, because 1+19=20. Similarly, the integers from 200 through 299 yield 20 '1's, and so forth. There are nine groups of integers (100-199, 200-299, …, 900-999), each yielding 20 '1's in the tens and ones places, so there are a total of 9(20)=180 '1's in the tens and ones places. Therefore, the 3-digit case yields 280 '1's, because 100+180=280.
• 4-digit case: 1000 has one '1'.
• Finally, by adding the numbers of '1's from the four aforementioned cases, 1+19+280+1=301.

Answer: 301

Challenge / Extension: What if we considered all the integers from 1 to 1,000,000 inclusive? The answer is 600,001. Try to see why. It will help to use recursive reasoning, in which you build up on smaller cases. Explanation 1 probably provides the easier template.